2810. Proposed by H. S. UHLER, Yale University.

In the expansion of the following determinant or eliminant, find the total number of terms, and the number of terms having the coefficients $+1$, -1, $+2$, -2, $+3$, -3, $+4$, -4, $+5$, -5, $+6$, -8, $+10$, respectively.

$$\begin{vmatrix} a & b & c & d & e & 0 & 0 & 0 \\ 0 & a & b & c & d & e & 0 & 0 \\ 0 & 0 & a & b & c & d & e & 0 \\ 0 & 0 & 0 & a & b & c & d & e \\ A & B & C & D & E & 0 & 0 & 0 \\ 0 & A & B & C & D & E & 0 & 0 \\ 0 & 0 & A & B & C & D & E & 0 \\ 0 & 0 & 0 & A & B & C & D & E \end{vmatrix}$$

2811. Proposed by J. L. RILEY, Stephenville, Texas.

Given the cube roots of 60, 61, 63, and 64, to find the cube root of 62 by the method of differences.

2812. Proposed by C. N. SCHMALL, New York City.

If $F(x, y, z)$ be a homogeneous function of x, y, z, which becomes $\phi(u, v, w)$ by the elimination of x, y, z, by means of the equations $\partial F/\partial x = u$, $\partial F/\partial y = v$, $\partial F/\partial z = w$; show that

$$\frac{\partial F}{\partial u}\Big/x = \frac{\partial F}{\partial v}\Big/y = \frac{\partial F}{\partial w}\Big/z.$$

2813. Proposed by PAUL CAPRON, U. S. Naval Academy.

An ellipse having the major-axis $2a$ and the eccentricity ϵ, is revolved first about its major axis, forming a prolate spheroid, then about its minor axis forming an oblate spheroid. Show that the surfaces of these spheroids are, respectively,

$$2\pi a^2(1/\epsilon\sqrt{1-\epsilon^2}\sin^{-1}\epsilon + 1)$$

and

$$2\pi a^2\left[2 + 1/\epsilon(1-\epsilon^2)\log\left(\frac{1+\epsilon}{1-\epsilon}\right)\right].$$

SOLUTIONS OF PROBLEMS.

339 (Calculus) [June, 1913; May, 1919]. Proposed by T. H. GRONWALL, Washington, D. C.

To show that for any real value of x

$$\left|\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)\right| \leqq \frac{1}{n+1}, \quad \text{and} \quad \left|\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right)\right| \leqq \frac{1}{n+1}.$$

I. Solution by Otto Dunkel, Washington University.

The function $y = \sin x/x$, $x \neq 0$; $y = 1$, $x = 0$, is single-valued and continuous for all values of x and it can be expressed as a power series

$$y = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots.$$

The power series shows that the function possesses all of its derivatives at $x = 0$, and it is easily seen from the power series or from the original form of the function that the derivatives exist for all other values of x. Hence, we may write the sequence of equations for finding the derivatives

$$xy = \sin x, \qquad y + xy' = \sin\left(\frac{\pi}{2} + x\right),$$

$$2y' + xy'' = \sin(\pi + x), \qquad \cdots, \qquad (n+1)y^{(n)} + xy^{(n+1)} = \sin\left(\frac{n+1}{2}\pi + x\right), \qquad \cdots$$

These equations, taken in turn, show that y, y', y'', $\cdots$, approach zero as x becomes infinite Thus $y^{(n)}$, for example, attains its maximum or minimum value for some finite value of x. Fo such a value of x we have $y^{(n+1)} = 0$ and

$$(n+1)y^{(n)} = \sin\left(\frac{n+1}{2}\pi + x\right)$$

and hence

$$|y^{(n)}| \leqq \frac{1}{n+1}.$$

The proof of the second inequality may be carried through in the same manner.

II. Solution by H. S. Uhler, Yale University.

Let $r \equiv (\sin x)/x$ and $s \equiv \sin x$, then

$$r = \frac{1}{x}\cdot s, \tag{0}$$

$$\frac{dr}{dx} + \frac{1}{x}\cdot r = \frac{1}{x}\cdot\frac{ds}{dx}, \tag{1}$$

$$\frac{d^2r}{dx^2} + \frac{2}{x}\cdot\frac{dr}{dx} = \frac{1}{x}\cdot\frac{d^2s}{dx^2}, \tag{2}$$

$$\cdots\cdots\cdots$$

$$\frac{d^{n-1}r}{dx^{n-1}} + \frac{n-1}{x}\cdot\frac{d^{n-2}r}{dx^{n-2}} = \frac{1}{x}\cdot\frac{d^{n-1}s}{dx^{n-1}}, \tag{$n-1$}$$

$$\frac{d^nr}{dx^n} + \frac{n}{x}\cdot\frac{d^{n-1}r}{dx^{n-1}} = \frac{1}{x}\cdot\frac{d^ns}{dx^n}. \tag{n}$$

In order to find an explicit formula for the general derivative of r with respect to x in terms of n, x, $\sin x$, and $\cos x$ we may proceed as follows:

Case 1. *n even.* Multiply equations (n), $(n-1)$, $(n-2)$, $(n-3)$, $\cdots$, (2), (1), (0) by $+1$ $-\frac{n}{x}$, $+\frac{n}{x}\cdot\frac{n-1}{x}$, $-\frac{n}{x}\cdot\frac{n-1}{x}\cdot\frac{n-2}{x}$, $\cdots$, $+\frac{n}{x}\cdot\frac{n-1}{x}\cdots\frac{4}{x}\cdot\frac{3}{x}$, $-\frac{n}{x}\cdot\frac{n-1}{x}\cdots\frac{3}{x}\cdot\frac{2}{x}$, $+\frac{n}{x}\cdot\frac{n-1}{x}\cdots\frac{2}{x}\cdot\frac{1}{x}$, respectively, and add the results. We then find

$$\frac{d^nr}{dx^n} = \pm\frac{1}{x}\cdot F_1(x)\cdot\sin x \pm \frac{n}{x^2}\cdot F_2(x)\cdot\cos x, \tag{A}$$

where

$$F_1(x) \equiv 1 - \frac{n(n-1)}{x^2} + \frac{n(n-1)(n-2)(n-3)}{x^4} - \cdots \mp \frac{n(n-1)\cdots 4\cdot 3}{x^{n-2}} \pm \frac{n(n-1)\cdots 2\cdot 1}{x^n},$$

$$F_2(x) \equiv 1 - \frac{(n-1)(n-2)}{x^2} + \frac{(n-1)(n-2)(n-3)(n-4)}{x^4} - \cdots \pm \frac{(n-1)(n-2)\cdots 5\cdot 4}{x^{n-4}} \mp \frac{(n-1)(n-2)\cdots 3\cdot 2}{x^{n-2}}$$

The upper or lower signs are to be used throughout according as n is of the form $4m$ or $4m-2$, respectively; $m = 1, 2, 3, \cdots$. When $n = 4m$, the number of terms in $F_1(x)$ and $F_2(x)$ is $2m+1$ and $2m$, whereas when $n = 4m-2$ the number of terms in $F_1(x)$ and $F_2(x)$ is $2m$ and $2m-1$, respectively.

Case 2. *n odd.* Multiply equations (n), $(n-1)$, $\cdots$, (1), (0) by the same factors as were used above, observing that the signs of the last three factors will now be $-$, $+$, $-$; and add.

$$\frac{d^nr}{dx^n} = \mp\frac{1}{x}\cdot F_3(x)\cdot\cos x \pm \frac{n}{x^2}\cdot F_4(x)\cdot\sin x \tag{B}$$

where

$$F_3(x) \equiv 1 - \frac{n(n-1)}{x^2} + \frac{n(n-1)(n-2)(n-3)}{x^4} - \cdots \pm \frac{n(n-1)\cdots 5\cdot 4}{x^{n-3}} \mp \frac{n(n-1)\cdots 3\cdot 2}{x^{n-1}},$$

$$F_4(x) \equiv 1 - \frac{(n-1)(n-2)}{x^2} + \frac{(n-1)(n-2)(n-3)(n-4)}{x^4} - \cdots \pm \frac{(n-1)(n-2)\cdots 4\cdot 3}{x^{n-3}} \mp \frac{(n-1)(n-2)\cdots 2\cdot 1}{x^{n-1}}.$$

The upper or lower signs are to be used throughout according as n is of the form $4m - 1$ or $4m - 3$, respectively. When $n = 4m - 1$, or $n = 4m - 3$ the number of terms in both $F_3(x)$ and $F_4(x)$ is $2m$ or $2m - 1$, respectively. In all cases, the total number of terms in the right hand members of formulæ (A) and (B) is $n + 1$.

The following properties of these formulæ will help to throw light on the general problem. Since $F_1(x)$, $F_2(x)$, $F_3(x)$, and $F_4(x)$ involve only even powers of x it is seen, at once, from the outstanding factors $\pm \frac{\sin x}{x}$, $\pm \frac{n \cos x}{x^2}$, and $\mp \frac{\cos x}{x}$, $\pm \frac{n \sin x}{x^2}$ that $d^n r/dx^n$, when numerically equal positive and negative values are substituted for x (n being always a positive integer), does not change sign for n even, and it does change sign for n odd. Hence the graph of $y = d^n r/dx^n$ is symmetrical with respect to the y-axis when n is even. On the other hand, the ordinates are equal in magnitude but opposite in sign for $x = +c$ and $x = -c$, when n is odd. When x is neither zero nor infinite it is evident that $d^n r/dx^n$ is finite, continuous, and single-valued. By reducing all terms of the right-hand members of (A) and (B) to the common denominator x^{n+1} we see that $d^n r/dx^n$ assumes the so-called indeterminate form $0/0$ when $x = 0$. Differentiating the re-written numerators of (A) and (B), and dividing each result by $d/dx(x^{n+1})$, *i.e.*, by $(n + 1)x^n$, we find $\pm \frac{x^n \cos x}{(n+1)x^n}$ and $\pm \frac{x^n \sin x}{(n+1)x^n}$, respectively. Cancelling x^n and then substituting zero for x we obtain $\pm \frac{1}{n+1}$ and 0 for the limits of $d^n r/dx^n$ corresponding, respectively, to even and odd values of n. Finally, since $|\sin x| \not> 1$, and $|\cos x| \not> 1$, (A) and (B) show that $d^n r/dx^n$ vanishes as x becomes infinite, n remaining finite. It has thus been demonstrated that, for all values of x, $d^n r/dx^n$ is a finite, continuous, and single-valued function of x.

The properties of $d^n r/dx^n$ just proved show that the greatest numerical values of this derivative may be obtained from its algebraic maxima and minima, if it possesses any. Accordingly the next step in the argument is to establish the existence of maxima and minima.

A necessary condition that $d^n r/dx^n$ shall have maxima and minima is $d^{n+1}r/dx^{n+1} = 0$. Hence it must be shown that the $(n + 1)$th derivative can always vanish when neither n nor x is infinite. When n is even, $n + 1$ is odd so that formula (B) applies. When n is odd, $n + 1$ is even and formula (A) applies. It is not necessary to write out separately and in detail the equations for $d^{n+1}r/dx^{n+1} = 0$ since both assume the form

$$f(x)\cdot \sin x = xF(x)\cdot \cos x \tag{C}$$

where $f(x)$ and $F(x)$ are polynomials in x^2, each having integral coefficients and a term free from x.

We are to show that the necessary condition (C) is an equation which always has real roots in addition to $x = 0$. Now $f(x) = 0$ and $F(x) = 0$ either have one or more real roots in common, or they have no common root. If such a root be admitted then a proof of the existence of a real root of condition (C) would be superfluous. Supposing that $f(x) = 0$ and $F(x) = 0$ have no common root we may demonstrate the existence of real roots of equation (C) in the following manner.

In the first place, assume that both $f(x) = 0$ and $F(x) = 0$ have real roots. Then take a positive value of x, R say, which is greater than the numerically largest root of $f(x) = 0$ and $F(x) = 0$. Also imagine the curves $y = f(x)\cdot \sin x$ and $y = xF(x)\cdot \cos x$ plotted to the same scale on the same diagram, and consider the possibility of the two loci having real intersections. For values of x between R and $+\infty$ the curves can cross the axis of x only when $\sin x$ and $\cos x$ vanish, for we have chosen R too large to permit either $f(x)$ or $F(x)$ to become zero. The curves $(x > R)$ intersect the x-axis at *alternate* points which have the constant interval $\pi/2$. In short, for $x > R$, the loci have the general nature of $y = \sin x$ and $y = \cos x$ except in so far as each trigonometric ratio is multiplied by a function of x that is finite, continuous, and single-valued ("damped" harmonic curves). It is clear, therefore, that for $x > R$ the loci intersect each other in an infinite number of discrete real points and hence that equation (C) has an infinite number of real roots.

The same argument applies if $f(x) = 0$ and $F(x) = 0$ have no real roots. In this case, however, we do not have to bother about R, as the possibility of one locus avoiding the other by crossing and recrossing the x-axis at the vanishing points of $f(x)$ and $F(x)$ no longer obtains.

For the sake of completeness the following comments seem appropriate. The question as to whether $f(x) = 0$ and $F(x) = 0$ have one or more common roots is not germane to the proof just presented. A similar remark applies to a discussion of the nature of the roots of these equations taken separately. Again, condition (C) cannot be fulfilled by the simultaneous vanishing of the members of the pairs $f(x)$, $\cos x$ and $F(x)$, $\sin x$. For, $\cos x$ and $\sin x$ vanish when, and only ($x \neq 0$) when x is an integral multiple of $\pm (\pi/2)$ or $\pm \pi$, and it is a well-established fact,—upon which the proofs of the transcendentality of π depend (Lindemann, Hilbert),—that π cannot be a root of any "algebraic" equation, *a fortiori* for one having integral coefficients, *e.g.*, $f(x) = 0$ and $F(x) = 0$.

Having now established the existence of real roots of (C) we must next demonstrate the sufficiency of this condition, that is, we must show that $d^{n+2}r/dx^{n+2}$ cannot vanish when $d^{n+1}r/dx^{n+1} = 0$. The $(n+2)$th equation of the set given at the very beginning is

$$\frac{d^{n+2}r}{dx^{n+2}} + \frac{n+2}{x} \cdot \frac{d^{n+1}r}{dx^{n+1}} = \frac{1}{x} \cdot \frac{d^{n+2}s}{dx^{n+2}}.$$

Therefore, when $d^{n+1}r/dx^{n+1} = 0$, we have also

$$\frac{d^{n+2}r}{dx^{n+2}} = \frac{1}{x} \cdot \frac{d^{n+2}s}{dx^{n+2}},$$

where $d^{n+2}s/dx^{n+2}$ has the values $-\sin x$, $+\cos x$, $+\sin x$, $-\cos x$ corresponding, respectively, to $n = 4m$, $4m - 1$, $4m - 2$, $4m - 3$. Now $(\cos x)/x$ cannot here vanish for a finite value of x because the necessary condition (C) is not fulfilled (as explained above) when $\cos x = 0$. Again $(\sin x)/x$ cannot vanish for a non-infinite value of x since, as before, condition (C) prevents $\sin x = 0$ for $x \neq 0$, and the ratio $(\sin x)/x$ approaches 1 for $x \doteq 0$. Consequently it has been shown that the necessary condition for the occurrence of maxima and minima of $d^n r/dx^n$ is also sufficient.

We are now prepared to discuss the values of $d^n r/dx^n$ at its maxima and minima. The $(n+1)$th equation of the original list would be

$$\frac{d^{n+1}r}{dx^{n+1}} + \frac{n+1}{x} \cdot \frac{d^n r}{dx^n} = \frac{1}{x} \cdot \frac{d^{n+1}s}{dx^{n+1}}.$$

Hence, when $d^{n+1}r/dx^{n+1} = 0$ we have also

$$\frac{d^n r}{dx^n} = \frac{1}{n+1} \cdot \frac{d^{n+1}s}{dx^{n+1}},$$

where $d^{n+1}s/dx^{n+1}$ has the values $+\cos x$, $+\sin x$, $-\cos x$, $-\sin x$ associated, respectively, with $n = 4m$, $4m - 1$, $4m - 2$, $4m - 3$. Therefore, $| d^n r/dx^n |$ equals $1/(n+1) \cdot | \cos x |$ or $1/(n+1) \cdot | \sin x |$ when n is even or odd, respectively. Since $| \cos x | \leqq 1$ and $| \sin x | \leqq 1$ it follows that $| d^n r/dx^n |$ cannot exceed $1/(n+1)$. As we have seen that (C) is satisfied by $x = 0$ but not by any integral multiple of either $\pm (\pi/2)$ or $\pm \pi$, the final results of the analysis may be written:

$$\left| \frac{d^n r}{dx^n} \right|_{x=0} = \frac{1}{n+1}, \text{ for } n \text{ even},$$

$$\left| \frac{d^n r}{dx^n} \right| < \frac{1}{n+1}, \text{ for } n \text{ odd, or for } n \text{ even and } x \neq 0.$$

REMARK. In the special case $n = 1$, I have followed the numerical values of

$$\frac{dr}{dx} = \frac{x \cos x - \sin x}{x^2}$$

from $x = 0$ through its greatest arithmetic maximum, which occurs when $x = \pm 2.081576\cdots$ (or $119° \, 15' \, 55.87'' \cdots$). The corresponding value of $| dr/dx |$ is $0.436182\cdots$, which is about 12.8 per cent. less than $1/(n+1) = 1/2$.

Attention must next be turned to

$$\frac{1 - \cos x}{x} \equiv u.$$

Writing c in place of $1 - \cos x$ the general equation is found at once to be

$$\frac{d^n u}{dx^n} + \frac{n}{x} \cdot \frac{d^{n-1}u}{dx^{n-1}} = \frac{1}{x} \cdot \frac{d^n c}{dx^n}.$$

Using precisely the same multipliers and method as before, we obtain

$$\frac{d^n u}{dx^n} = \mp \frac{1}{x} \cdot F_1(x) \cdot \cos x \pm \frac{n}{x^2} \cdot F_2(x) \cdot \sin x + \frac{\underline{|n}}{x^{n+1}}, \tag{A'}$$

$$\frac{d^n u}{dx^n} = \mp \frac{1}{x} \cdot F_3(x) \cdot \sin x \mp \frac{n}{x^2} \cdot F_4(x) \cdot \cos x - \frac{\underline{|n}}{x^{n+1}}. \tag{B'}$$

When n is even, (A'), and x is given pairs of values $+ a$ and $- a$, $d^n u/dx^n$ assumes values that are equal in magnitude but opposite in sign. On the other hand, when n is odd this derivative alters neither in sign nor in magnitude when x is changed from $+ a$ to $- a$. Hence, when n is odd $y = d^n u/dx^n$ is symmetrical with respect to the axis of ordinates.

Since all the separate terms of formulæ (A') and (B') involve either $\sin x$, or $\cos x$, or $\underline{|n}$ in the numerator and some positive integral power of x in the denominator we see that $d^n u/dx^n$ is finite for all values of x between 0 and ∞. When x becomes infinite $d^n u/dx^n = 0$. When $x = 0$ the indeterminate form 0/0, when treated in the usual way, gives $\pm \frac{x^n \sin x}{(n + 1)x^n}$ for n even, and $\mp \frac{x^n \cos x}{(n + 1)x^n}$ for n odd. Therefore, for $x = 0$, the nth derivative of u is equal to 0 or $\mp \frac{1}{n + 1}$ according as n is even or odd, respectively.

For sake of brevity and variety, the question of the existence of maxima and minima may be settled by making use of the elementary properties of plane curves. The fraction c/x is finite, continuous, single-valued, with an infinite number of maxima and minima. Whenever u has a maximum or a minimum du/dx vanishes. In other words, du/dx changes sign for every value of x that corresponds to a maximum or a minimum of u. Also the general formulæ (A') and (B') show that any derivative is finite, continuous, and single-valued. Therefore, du/dx must likewise have an infinite number of maxima and minima. We then apply the foregoing argument to du/dx and d^2u/dx^2, and so on indefinitely. Obviously the same line of reasoning could have been applied to r and all of its x-derivatives. The earlier proof was presented *in extenso* because of its rigor and probable instructive value.

Proceeding as in the case of r, we find that, when $d^{n+1}u/dx^{n+1} = 0$,

$$\frac{d^n u}{dx^n} = \frac{1}{n + 1} \cdot \frac{d^{n+1} c}{dx^{n+1}}.$$

Accordingly, at a maximum or minimum,

$$\left|\frac{d^n u}{dx^n}\right| \text{ equals } \frac{1}{n + 1} \cdot |\sin x| \qquad \text{or} \qquad \frac{1}{n + 1} \cdot |\cos x|$$

according as n is even or odd. Finally we conclude that

$$\left|\frac{d^n u}{dx^n}\right|_{x=0} = \frac{1}{n + 1}, \text{ for } n \text{ odd},$$

$$\left|\frac{d^n u}{dx^n}\right| < \frac{1}{n + 1}, \text{ for } n \text{ even, or for } n \text{ odd and } x \neq 0.$$

2732 [1918, 444]. Proposed by PAUL CAPRON, U. S. Naval Academy.

A conical cup, filled with fluid, stands with the vertex upward on a smooth horizontal surface. The inner and outer surfaces of the cup are similar cones of revolution, having altitudes h and $h(1 + x)$; the ratio of the specific weights of the material of the cone and the fluid is σ; the height of a barometer column of the fluid is h_0. Show that for equilibrium

$$\frac{h_0}{h} (1 + x)^2 + \sigma x(1 + x + x^2/3) \nless 2/3.$$